

**CS 103ACE
Midterm 1
Review**

	To prove that this is true...	If you assume this is true...
$\forall x. A$	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .	Initially, do nothing . Once you find a z through other means, you can state it has property A .
$\exists x. A$	Find an x where A is true. Then prove that A is true for that specific choice of x .	Introduce a variable x into your proof that has property A .
$A \rightarrow B$	Assume A is true, then prove B is true.	Initially, do nothing . Once you know A is true, you can conclude B is also true.
$A \wedge B$	Prove A . Then prove B .	Assume A . Then assume B .
$A \vee B$	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. <i>(Why does this work?)</i>	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \rightarrow B$ and $B \rightarrow A$.	Assume $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Lecture 0: Set Theory Symbols

- \in & \notin Element of / not an element of
- \subseteq Subset of

- \emptyset Empty set
- $\{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$ Example of set-builder notation

- $A \cup B$ Set union
- $A \cap B$ Set intersection
- $A - B$ Set difference
- $A \Delta B$ Set symmetric difference

- $|S|$ Cardinality
- $\wp(S)$ Power set of S
- \aleph_0 Aleph-zero (the cardinality of \mathbb{N})

Lecture 1 / Problem Set 1 Definitions

Sets of numbers

- \mathbb{N} is the set of **natural numbers**: $\{0, 1, 2, 3, \dots\}$
- \mathbb{Z} is the set of **integers**: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- \mathbb{R} is the set of **real numbers**

Parity

- An integer n is called **even** if there is an integer k where $n = 2k$.
- An integer n is called **odd** if there is an integer k where $n = 2k + 1$.

Modular congruence

- For integers a and b , we say that $a \equiv_k b$ if there is an integer q such that $a = b + kq$.

Lecture 1: Proof Strategies

- A **universally-quantified statement** is a statement of the form

For all x , [some-property] holds for x .

- An **existentially-quantified statement** is a statement of the form

There is some x where [some-property] holds for x .

- To **directly prove a universally quantified statement**, allow the reader to pick an x and show that it has [some-property].
- To **directly prove an existentially quantified statement**, specifically say what to pick for x and show that it has [some-property].

See also the [Guide to Proofs](#)

Lecture 2: Definitions

- A **proposition** is a statement that is either true or false.
- The **negation** of a proposition X is a proposition that is true when X is false and is false when X is true
 - The negation of a universal is an existential.
 - The negation of an existential is a universal.
 - The negation of an implication “if P , then Q ” is “ P and not Q .”
- An **implication** is a statement of the form “If P is true, then Q is true.” P is called the **antecedent** and Q is called the **consequent**.
- The **contrapositive** of the implication “If P is true, then Q is true” is the implication “If Q is false, then P is false.”
- A **biconditional** is a statement of the form “ P if and only if Q ”

Lecture 2: Proof Strategies

See the [Guide to Proofs](#)

- **Proving an implication:** First, assume the antecedent. Then, prove the consequent.
- **Proof by contradiction:** First, assume that P is false. The goal is to show that this assumption is silly. Next, show this leads to an impossible result. Finally, conclude that since P can't be false, we know that P must be true.
- **Proof by contrapositive:** Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect. Then, explicitly state the contrapositive of what we want to prove. Then, go prove the contrapositive. (See “proving an implication”!)

Lecture 3: Propositional Connectives

Expression	English Translation	Negation of Expression
$A \wedge B$	A and B	2 possibilities: $\neg A \vee \neg B$, $A \rightarrow \neg B$
$A \vee B$	A or B	$\neg A \wedge \neg B$
$\neg A$	Not A	A
$A \rightarrow B$	A implies B	$A \wedge \neg B$
$A \leftrightarrow B$	A if and only if B	2 possibilities: $A \leftrightarrow \neg B$, $\neg A \leftrightarrow B$
\top	True	\perp
\perp	False	\top

Truth table tool

Lecture 3: Operators

Operator precedence:

- \neg binds to whatever immediately follows it.
- \wedge and \vee bind more tightly than \rightarrow : $p \wedge q \rightarrow r$ is equivalent to $(p \wedge q) \rightarrow r$
- Operators are right-associative: $p \rightarrow q \rightarrow r$ is equivalent to $p \rightarrow (q \rightarrow r)$

De Morgan's Laws:

- $\neg(p \wedge q)$ is the same as $\neg p \vee \neg q$
- $\neg(p \vee q)$ is the same as $\neg p \wedge \neg q$

Lecture 4: Quantifiers

- \exists is the **existential quantifier**. A statement of the form

$$\exists x. \textit{some-formula}$$

is true for a certain world if there exists a choice of x where *some-formula* is true when that x is plugged into it.

Existentially-quantified statements are false unless there's a positive example. The \exists quantifier usually is paired with \wedge .

- \forall is the **universal quantifier**. A statement of the form

$$\forall x. \textit{some-formula}$$

is true for a certain world if, for every choice of x , the statement *some-formula* is true when x is plugged into it. Universally-quantified statements are true unless there's a counterexample. The \forall quantifier usually is paired with \rightarrow .

Lecture 4: Type-Checking Table

	... operate on and produce
Connectives (\leftrightarrow , \wedge , etc.) ...	propositions	a proposition
Predicates ($=$, etc.) ...	objects	a proposition
Functions ...	objects	an object

Lecture 5: First Order Logic

“All A s are B s”

$$\forall x. (A(x) \rightarrow B(x))$$

“Some A s are B s”

$$\exists x. (A(x) \wedge B(x))$$

“No A s are B s”

$$\forall x. (A(x) \rightarrow \neg B(x))$$

“Some A s aren't B s”

$$\exists x. (A(x) \wedge \neg B(x))$$

Lecture 5: First Order Logic

	When is this true?	When is this false?
$\forall x. P(x)$	For all objects x , $P(x)$ is true.	$\exists x. \neg P(x)$
$\exists x. P(x)$	There is an x where $P(x)$ is true.	$\forall x. \neg P(x)$
$\forall x. \neg P(x)$	For all objects x , $P(x)$ is false.	$\exists x. P(x)$
$\exists x. \neg P(x)$	There is an x where $P(x)$ is false.	$\forall x. P(x)$

Lecture 5: Quantifiers over Sets

The notation

$$\forall x \in S. P(x)$$

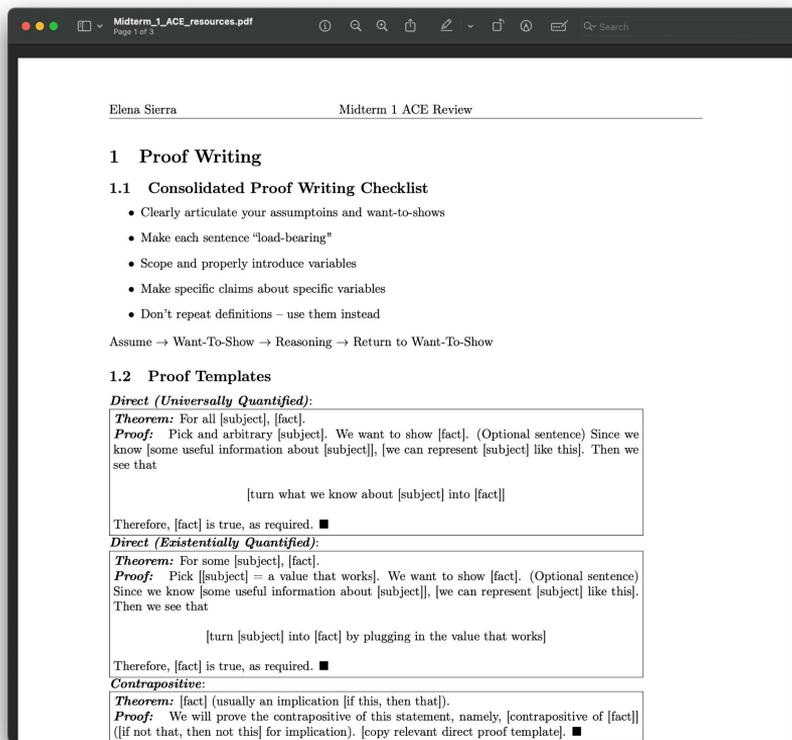
means “for any element x of set S , $P(x)$ holds.” (It’s vacuously true if S is empty.)

The notation

$$\exists x \in S. P(x)$$

means “there is an element x of set S where $P(x)$ holds.” (It’s false if S is empty.)

Elena's Consolidated Tips



Practice Problems

- **Multimodular Congruence**
 - Midterm 5, problem 3
 - Assuming vs proving existentially-quantified statements
- **Translating into Logic, Take Two**
 - Midterm 4, problem 2
 - Logic translations
- **Odd Modular Congruence**
 - Midterm 4, problem 4
 - Parity & proof writing
- **Reverse Executable Set Theory**
 - Midterm 6, problem 2
 - Interpreting code and set notation
- **Autocontrapositives**
 - Midterm 2, problem 1
 - Negation and translating statements



Multimodular Congruence

Midterm 5 Problem 3

Prove that for any integers x and y , we have $x \equiv_{6,10} y$ if and only if $x \equiv_2 y$.

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1. Translate to first order logic

$$\forall x \in \mathbb{Z}. \forall y \in \mathbb{Z} (x \equiv_{6,10} y \leftrightarrow x \equiv_2 y)$$

Prove that for any integers x and y , we have $x \equiv_{6,10} y$ if and only if $x \equiv_2 y$.

1. Translate to first order logic

$$\forall x \in \mathbb{Z}. \forall y \in \mathbb{Z}$$

$$(x \equiv_{6,10} y \iff x \equiv_2 y)$$

2. Move outside to inside with the table

	If you <i>assume</i> this is true...	To <i>prove</i> that this is true...
$\forall x. A$	Initially, <i>do nothing</i> . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
$\exists x. A$	Introduce a variable x into your proof that has property A .	Find an x where A is true. Then prove that A is true for that specific choice of x .
$A \rightarrow B$	Initially, <i>do nothing</i> . Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \wedge B$	Assume A . Also assume B .	Prove A . Also prove B .
$A \vee B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. <i>(Why does this work?)</i>
$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Prove that for any integers x and y , we have $x \equiv_{6,10} y$ if and only if $x \equiv_2 y$.

1. Translate to first order logic

$$\forall x \in \mathbb{Z}. \forall y \in \mathbb{Z}$$

$$(x \equiv_{6,10} y \iff x \equiv_2 y)$$

2. Move outside to inside with the table

3. Write first two sentences

Pick any arbitrary integers x and y . We want to show that $x \equiv_{(6,10)} y$ if and only if $x \equiv_{(2)} y$.

	If you <i>assume</i> this is true...	To <i>prove</i> that this is true...
$\forall x. A$	Initially, <i>do nothing</i> . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
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Pick any arbitrary integers x and y . We want to show that $x \equiv_{(6,10)} y$ if and only if $x \equiv_{(2)} y$.

1. Move in a layer in first order logic

$$(x \equiv_{6,10} y \leftrightarrow x \equiv_a y)$$

2. Move outside to inside with the table

3. Structure both directions of biconditional

We need to prove both directions of the implication.

First, assume $x \equiv_{(6,10)} y$. We need to show that there exists some integer k where $x = y + 2k$.

...

Second, assume $x \equiv_{(2)} y$. We need to show that there exists some integers a and b where $x = y + 6a + 10b$.

	If you assume this is true...	To prove that this is true...
$\forall x. A$	Initially, do nothing . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
$\exists x. A$	Introduce a variable x into your proof that has property A .	Find an x where A is true. Then prove that A is true for that specific choice of x .
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Pick any arbitrary integers x and y . We want to show that $x \equiv_{(6,10)} y$ if and only if $x \equiv_{(2)} y$.

1. Move in a layer in first order logic

$$(x \equiv_{6,10} y \leftrightarrow x \equiv_a y)$$

WARNING! We will need to introduce x and y differently for each direction

2. Move outside to inside with the table

3. Structure both directions of biconditional

We need to prove both directions of the implication.

First, assume $x \equiv_{(6,10)} y$. We need to show that there exists some integer k where $x = y + 2k$.

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Second, assume $x \equiv_{(2)} y$. We need to show that there exists some integers a and b where $x = y + 6a + 10b$.

	If you assume this is true...	To prove that this is true...
$\forall x. A$	Initially, do nothing . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
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$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

First we will prove that if $x \equiv_{(6,10)} y$ then $x \equiv_{(2)} y$.

1. Start with first implication

$$x \equiv_{(6,10)} y \rightarrow x \equiv_{(2)} y$$

2. Move outside to inside with the table

	If you assume this is true...	To prove that this is true...
$\forall x. A$	Initially, do nothing . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
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$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

First we will prove that if $x \equiv_{(6,10)} y$ then $x \equiv_{(2)} y$. Pick some arbitrary integers x and y where $x \equiv_{(6,10)} y$.

1. Start with first implication

$$x \equiv_{6,10} y \rightarrow x \equiv_a y$$

2. Move outside to inside with the table

3. Write assumption

Since, $x \equiv_{(6,10)} y$, we know there are integers q and r where $x = y + 6q + 10r$.

	If you assume this is true...	To prove that this is true...
$\forall x. A$	Initially, do nothing . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
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$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
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First we will prove that if $x \equiv_{(6,10)} y$ then $x \equiv_{(2)} y$. Pick some arbitrary integers x and y where $x \equiv_{(6,10)} y$. Since, $x \equiv_{(6,10)} y$, we know there are integers q and r where $x = y + 6q + 10r$.

1. Start with first implication

$$x \equiv_{6,10} y \rightarrow x \equiv_a y$$

2. Move outside to inside with the table

3. Write want-to-show, and show it.

Pick an $k = (3q + 5r)$. Then we see that $x = y + 6q + 10r = y + 2(3q + 5r) = y + 2k$. Therefore, $x \equiv_{(2)} y$, as required.

	If you assume this is true...	To prove that this is true...
$\forall x. A$	Initially, do nothing . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
$\exists x. A$	Introduce a variable x into your proof that has property A .	Find an x where A is true. Then prove that A is true for that specific choice of x .
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$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next, we'll prove that if $x \equiv(2) y$, then $x \equiv(6,10) y$.

1. Second implication

$$x \equiv_a y \rightarrow x \equiv_{6,10} y$$

2. Move outside to inside with the table

	If you assume this is true...	To prove that this is true...
$\forall x. A$	Initially, do nothing . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
$\exists x. A$	Introduce a variable x into your proof that has property A .	Find an x where A is true. Then prove that A is true for that specific choice of x .
$A \rightarrow B$	Initially, do nothing . Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \wedge B$	Assume A . Also assume B .	Prove A . Also prove B .
$A \vee B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. <i>(Why does this work?)</i>
$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next, we'll prove that if $x \equiv(2) y$, then $x \equiv(6,10) y$. Pick some arbitrary integers x and y where $x \equiv(2) y$.

1. Second implication

$$x \equiv_2 y \rightarrow x \equiv_{6,10} y$$

2. Move outside to inside with the table

3. Write out assumption

Since, $x \equiv(2) y$, we know there is some integer k where $x = y + 2k$.

	If you assume this is true...	To prove that this is true...
$\forall x. A$	Initially, do nothing . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
$\exists x. A$	Introduce a variable x into your proof that has property A .	Find an x where A is true. Then prove that A is true for that specific choice of x .
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$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next, we'll prove that if $x \equiv(2) y$, then $x \equiv(6,10) y$. Pick some arbitrary integers x and y where $x \equiv(2) y$. Since, $x \equiv(2) y$, we know there is some integer k where $x = y + 2k$.

1. Second implication

$$x \equiv_2 y \rightarrow x \equiv_{6,10} y$$

2. Move outside to inside with the table

3. Write want-to-show, and show it.

Pick an $q = 2k$ and $r = -k$. Then we see that $y + 6q + 10r = y + 6(2k) + 10(-k) = y + 12k - 10k = y + 2k = x$. Therefore, $x \equiv(6,10) y$, as required.

	If you assume this is true...	To prove that this is true...
$\forall x. A$	Initially, do nothing . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
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$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Finally: Combine our steps & edit our writing

Pick any arbitrary integers x and y . We want to show that $x \equiv_{(6,10)} y$ if and only if $x \equiv_{(2)} y$. We need to prove both directions of the implication.

First, assume $x \equiv_{(6,10)} y$. We need to show that there exists some integer k where $x = y + 2k$.

Since, $x \equiv_{(6,10)} y$, we know there are integers q and r where $x = y + 6q + 10r$. Pick an $k = (3q + 5r)$. Then we see that $x = y + 6q + 10r = y + 2(3q + 5r) = y + 2k$. Therefore, $x \equiv_{(2)} y$, as required.

Second, assume $x \equiv_{(2)} y$. We need to show that there exists some integers a and b where $x = y + 6a + 10b$.

Since, $x \equiv_{(2)} y$, we know there is some integer c where $x = y + 2c$. Pick $a = 2c$ and $b = -c$. Then we see that $y + 6a + 10b = y + 6(2c) + 10(-c) = y + 12c - 10c = y + 2c = x$. Therefore, $x \equiv_{(6,10)} y$, as required.